Matrices and Augmented Matrices

Finite Math

10 March 2017

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$$A = \begin{bmatrix} 1 & -4 & 5 \\ 7 & 0 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} -4 & 5 & 12 & 4 \\ 0 & 1 & 8 & 3 \\ -3 & 0 & 9 & 0 \\ 7 & -9 & 22 & 10 \end{bmatrix}$$

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A matrix is called an $m \times n$ matrix if it has m rows and n columns. The expression $m \times n$ is called the size of the matrix. The numbers m and n are called the dimensions of the matrix. If $m = n$, the matrix is called a square matrix. A matrix with only 1 column is called a column matrix and a matrix with only 1 row is called a row matrix.

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For example, the matrix A above is a 2×3 matrix and the matrix B is a 4×4 matrix and so B is a square matrix.

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When we write an arbitrary matrix we use the *double subscript notation*, a_{ij}, which is read as “a sub i-j”, for example, the element a_{23} is read as “a sub two-three” (not as “a sub twenty-three”); sometimes we will drop “sub” and just say “a two-three”.

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$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

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The *principal diagonal* (or main diagonal) of a matrix is the diagonal formed by the elements $a_{11}, a_{22}, a_{33}, \ldots$. This diagonal always starts in the upper left corner, but it doesn't have to end in the bottom right.

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Augmented Matrices

In this section, we will stick with systems of 2 equations. Given a system of equations

$$\begin{array}{rcrcl} a_{11}x_1 & + & a_{12}x_2 & = & k_1 \\ a_{21}x_1 & + & a_{22}x_2 & = & k_2 \end{array}$$

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and the *constant matrix*

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We can also put these two matrices together and form an *augmented matrix* associated to the system

$$\left[\begin{array}{cc:c} a_{11} & a_{12} & k_1 \\ a_{21} & a_{22} & k_2 \end{array}\right]$$

Augmented Matrices

Example

Find the augmented matrix associated to the system

$$\begin{array}{rcrcr} 3x & + & 4y & = & 1 \\ x & - & 2y & = & 7 \end{array}$$

Notation

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We will number the rows of a matrix from top to bottom and the columns of a matrix from left to right. When referring to the i^{th} row of a matrix we write R_i (for example R_2 refers to the second row) and we use C_j to refer to the j^{th} column.

Using Augmented Matrices

Definition (Row Equivalent)

We say that two augmented matrices are row equivalent if they are augmented matrices of equivalent linear systems. We write a $\sim$ *between two augmented matrices which are row equivalent.*

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This definition immediately leads to the following theorem

Using Augmented Matrices

Theorem

An augmented matrix is transformed into a row-equivalent matrix by performing any of the row operations:

(a) *Two rows are interchanged* ($R_i \leftrightarrow R_j$).

(b) *A row is multiplied by a nonzero constant* ($kR_i \rightarrow R_i$).

(c) *A constant multiple of one row is added to another row* ($kR_j + R_i \rightarrow R_i$).

The arrow $\rightarrow$ is used to mean "replaces."

Solving Linear Systems Using Augmented Matrices

When solving linear systems using augmented matrices, the goal is to use row operations as needed to get a 1 for every entry on the principal diagonal and zeros everywhere else on the left side of the augmented matrix. That is, the goal is to turn it into an augmented matrix of the form

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$$\left[\begin{array}{cc|c} 1 & 0 & m \\ 0 & 1 & n \end{array}\right]$$

which corresponds to the system

$$\begin{aligned} x \quad\quad &= m \\ y &= n \end{aligned}$$

thus telling us that $x = m$ and $y = n$.

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Solve the following system using an augmented matrix

$$\begin{array}{rcrcr} 3x & + & 4y & = & 1 \\ x & - & 2y & = & 7 \end{array}$$

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Solve the system using an augmented matrix

$$\begin{array}{rcrcl} 2x_1 & - & 3x_2 & = & 6 \\ 3x_1 & + & 4x_2 & = & \frac{1}{2} \end{array}$$

Now You Try It!

Example

Solve the system using an augmented matrix

$$\begin{array}{rcrcl} 5x & - & 2y & = & 11 \\ 2x & + & 3y & = & \frac{5}{2} \end{array}$$

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Example

Solve the system using an augmented matrix

$$\begin{array}{rcrcl} 5x & - & 2y & = & 11 \\ 2x & + & 3y & = & \frac{5}{2} \end{array}$$

Solution

$x = 2, y = -\frac{1}{2}$